OPERATIONS RESEARCH

Chapter 2

Transportation and Assignment Problems

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1.0 Introduction

In this chapter, we will discuss two special types of linear programming problem called transportation and assignment problems. Transportation problem (TP) has received this name because many of its applications involve determining how to transport goods in an optimal way. However, some of its important applications actually have nothing to do with transportation. For instance, consider production scheduling problem; its objective is to maximize the efficiency of the operation and reduce costs. On the other hand, assignment problem involves applications of assigning people to tasks. Although its applications appear to be quite different from those for the transportation, we see that the assignment problem can be viewed as a special type of transportation problem.

MODULE - 1: Mathematical Formulation and Initial BFS of Transportation Problem

1.1 Mathematical Formulation of TP

The transportation problem (TP) is concerned with determining an optimal strategy for transporting a commodity from a number of origins or sources to various destinations in such a way that the total transportation cost is minimized. Each origin has its own capacity or availability and each destination has its individual requirement.

		Destination				
		D_1	D_2	$\cdots$	D_n	Availability
Origin	O_1	c_{11}	c_{12}	$\cdots$	c_{1n}	a_1
	O_2	c_{21}	c_{22}	$\cdots$	c_{2n}	a_2
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
	O_m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	a_m
	Requirement	b_1	b_2	$\cdots$	b_n	

Table 1.1: Transportation table

Suppose that there are m origins O_i, $i = 1, 2, ..., m$ and n destinations D_j, $j = 1, 2, ..., n$ (Table 1.1). The ith origin's capacity (availability) is a_i units and the jth destination's requirement (demand) is b_j units. Let c_{ij} be the cost of shipping one unit of the commodity from the ith origin to the jth destination. If x_{ij} represents the number of units shipped from the ith source to the jth destination, the problem is to determine the transportation schedule so as to minimize the total transportation cost while satisfying the supply and demand conditions (rim conditions). Mathematically, a TP may be stated as follows:

$$\text{Minimize (total cost) } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{1.1}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, ..., m \quad (\textit{supply or availability constraints}) \tag{1.2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, ..., n \quad (\textit{demand or requirement constraints}) \tag{1.3}$$

$$x_{ij} \geq 0 \quad \text{for all } i \text{ and } j. \tag{1.4}$$

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ then the transportation problem is called balanced.

Theorem 1.1 (Existence of feasible solution): *A necessary and sufficient condition for the existence of a feasible solution of the TP* (1.1)-(1.4) *is*

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Proof: Part (i) (*The condition is necessary*)

Let there exist a feasible solution x_{ij} $(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$ of the TP (1.1)-(1.4). Then, we have

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}\right) = \sum_{i=1}^{m} b_j$$

Hence

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \lambda \text{ (say)}$$

Part (ii) (*The condition is sufficient*)

Suppose that there exists a feasible solution x_{ij} given by $x_{ij} = a_i b_j/\lambda$ for all i and j. Clearly, $x_{ij} \geq 0$ since $a_i > 0$, $b_j > 0$ for all i and j.

$$\text{Also,} \quad \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} (a_i b_j/\lambda) = \frac{a_i}{\lambda}\sum_{j=1}^{n} b_j = a_i, \quad i = 1, 2, ..., m$$

$$\text{and} \quad \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} (a_i b_j/\lambda) \frac{b_j}{\lambda}\sum_{i=1}^{m} a_i = b_j, \quad j = 1, 2, ..., n$$

Thus, x_{ij} satisfies all the constraints of the TP and hence it is a feasible solution.

Corollary 1.1: *There always exists an optimal solution of TP.*

Proof: Let us suppose that, for a TP, the relation $\sum_i a_i = \sum_j b_j$ holds so that a feasible solution x_{ij} exists. It follows from the constraints of the problem that each x_{ij} is bounded, viz., $0 \le x_{ij} \le \min(a_i,\ b_j)$. Thus the feasible region of the problem is closed, bounded and non-empty and hence there exists an optimal solution.

Theorem 1.2 (Basic feasible solution): *The number of basic variables in a TP with m origins and n destinations is at most $(m+n-1)$.*

Proof: Consider the following $(m+n)$ constraints of a balanced TP:

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,2,\ldots,m \tag{1.5}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,2,\ldots,n \tag{1.6}$$

Taking summation on both sides of (1.5), we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \textit{(balanced problem)}$$

Summing the first $(n-1)$ constraints of (1.6), we get

$$\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j$$

Subtracting the second summation from the former one, we get

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} &= \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j \\
\text{or, } \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}\right) &= b_n \\
\text{or, } \sum_{i=1}^{m} x_{in} &= b_n
\end{aligned}$$

which is precisely the nth constraint of (1.6). Thus one of the $(m+n)$ constraints is redundant and may be removed from the set of constraints. As a result, a basis consists of at most $(m+n-1)$ variables.

1.1.1 Loops in Transportation Table

In a transportation table, an ordered set of four or more cells is said to form a loop if

(i) any two adjacent cells in the ordered set lie in the same row (column);

(ii) not more than two adjacent cells in the ordered set lie in the same row (column);

(iii) the first and the last cells in the ordered set lie in the same row (column);

(iv) the ordered set must involve at least two rows (columns) of the table.

Loops are closed i.e. they have neither beginning nor end. Tables 1.2, 1.3 and 1.4 show three different loops while Table 1.5 shows a non-loop.

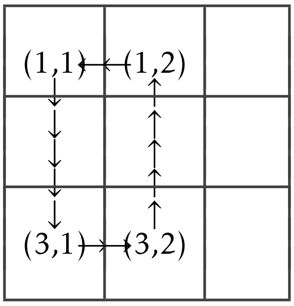

Table 1.2: Loop L_1

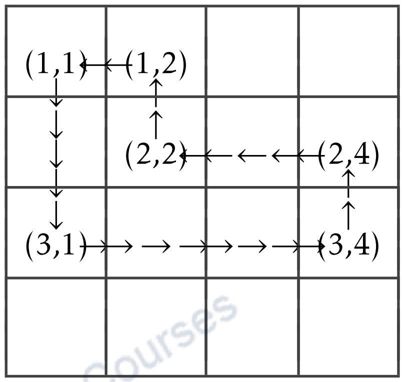

Table 1.3: Loop L_2

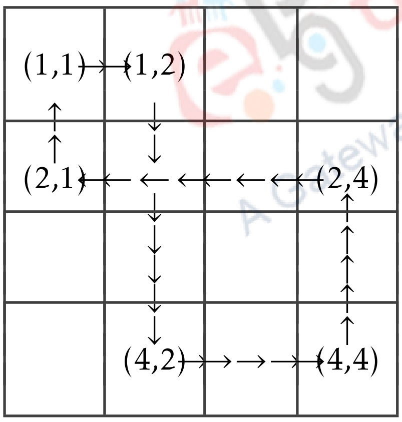

Table 1.4: Loop L_3

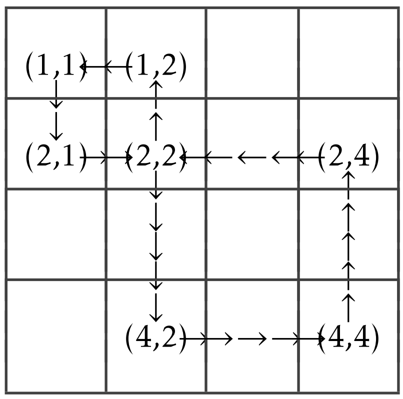

Table 1.5: Non-Loop L_4

Corollary 1.2: *A feasible solution of a TP is called basic if and only if the corresponding cells in the transportation table do not form a loop.*

1.2 Algorithm to Solve A TP

The solution of a TP may be summarized in the following steps :

Step 1: For the given TP, examine whether the total supply equals the total demand. If not, introduce a dummy row/column having all its cost elements zero, and supply/demand as positive difference of supply and demand.

Step 2: Find an initial BFS which must satisfy all the supply and demand conditions.

Step 3: Examine the solution for optimality, i.e., examine whether an unoccupied cell whose inclusion may result in an improved solution.

Step 4: If the solution is not optimal, modify the shipping schedule by including that unoccupied cell whose inclusion may result in an improved solution.

Step 5: Repeat Steps 3 and 4 until no further improvement is possible.

1.3 Method to Find An Initial BFS

There are several methods available to obtain an initial BFS for a TP:

1. North-West (N-W) Corner Rule

Steps involved in the N-W corner rule are given below:

Step 1: Make the maximum possible allocation to the upper left (north-west) corner cell in the first row depending upon the availability of supply for that row and demand requirement for the column containing that cell, i.e., min (a_1, b_1).

Step 2: Move to the next cell of the first row depending upon the remaining supply for that row and the demand requirement for the next column. Proceed till the row total is exhausted. There arise three possible cases to move to the next cell :

(i) $b_1 > a_1$: If the allocation made in step 1 is equal to the supply available at the first source (a_1, in the first row), then move vertically down to the cell (2, 1). Apply step 1 again, for the next allocation.

(ii) $b_1 < a_1$: If the allocation made in step 1 is equal to the demand of the first destination (b_1, in the first column) then move horizontally to the cell (1, 2). Apply step 1 again for the next allocation.

(iii) $b_1 = a_1$: If there is a tie then allocate $x_{11} = a_1 = b_1$ and move diagonally to the cell (2, 2).

Step 3: Repeat steps 1 and 2 moving down towards the lower/right (south-east) corner of the transportation table until all the requirements are satisfied.

Example 1.1: Obtain the initial BFS of the following TP using N-W corner rule.

		Destination			
		D_1	D_2	D_3	Capacity
Origin	O_1	5	4	3	100
	O_2	8	4	3	300
	O_3	9	7	5	300
	Requirement	300	200	200	

Table 1.6: Transportation Table

Solution: Since $\sum_i a_i = \sum_j b_j = 700$, the given TP is balanced and it has a feasible solution. Using N-W corner rule, the initial BFS is obtained (see Table 1.7) as $x_{11} = 100$, $x_{21} = 200$, $x_{22} = 100$, $x_{32} = 100$ and $x_{33} = 200$. The number of allocated cells is 5 which is equal to the value $m + n - 1 = 3 + 3 - 1 = 5$. Therefore, the solution is non-degenerate and the corresponding transportation cost is $Rs.(100 \times 5 + 200 \times 8 + 100 \times 4 + 100 \times 7 + 200 \times 5) = Rs.\ 4200$.

	D_1	D_2	D_3	a_i
O_1	[100] 5	4	3	100
O_2	[200] 8	[100] 4	3	300
O_3	9	[100] 7	[200] 5	300
b_j	300	200	200	

Table 1.7: Initial BFS using N-W corner rule

2. Least Cost Method or Matrix Minima Method

The steps involved in this method are as follows:

Step 1: Determine the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Let it be c_{ij}. Allocate x_{ij} =min (a_i, b_j) in the cell (i, j). If the smallest unit cost is not unique then select the cell where the maximum allocation can be made.

Step 2: If $x_{ij} = a_i$, cross off the ith row, decrease b_j by a_i and go to step 3.

If $x_{ij} = b_j$, cross off the jth column, decrease a_i by b_j and go to step 3.

If $x_{ij} = a_i = b_j$, cross off either the ith row or the jth column but not both.

Step 3: Repeat steps 1 and 2 until the resulting reduced transportation table satisfies all the requirements. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Example 1.2: Obtain the initial BFS of the following TP using Matrix minima method.

		Destination			
		D_1	D_2	D_3	Supply
Origin	O_1	16	20	12	200
	O_2	14	8	18	160
	O_3	26	24	16	90
	Demand	180	120	150	

Table 1.8: Transportation table for Example 1.2

Solution: The initial BFS as shown in Table 1.9 is $x_{11} = 50$, $x_{13} = 150$, $x_{21} = 40$, $x_{22} = 120$ and $x_{31} = 90$. Number of allocations $= 5 = m + n - 1$. The cost corresponding to this feasible solution is $Rs.(50 \times 16 + 150 \times 12 + 40 \times 14 + 120 \times 8 + 90 \times 26) = Rs.\ 6460$

	D_1	D_2	D_3	a_i
O_1	(50) 16	20	(150) 12	200
O_2	(40) 14	(120) 8	18	160
O_3	(90) 26	24	16	90
b_j	180	120	150	

Table 1.9: Initial solution using Matrix minima method

3. Row (Column) Minima Method

In this method, instead of finding the minimum cost cell, we find the minimum cost cell of the row (column). We give below the steps of row minima method:

Step 1: Determine the smallest cost in the first row of the transportation table. Let it be c_{1j}. Allocate as much as possible i.e., $x_{1j} = \min.\ (a_1, b_j)$ in the cell $(1, j)$.

Step 2: If $x_{1j} = a_1$, cross off the first row, decrease b_j by a_1 and go to step 3.
If $x_{1j} = b_j$, cross off the jth column, decrease a_1 by b_j and go to step 3.
If $x_{1j} = a_1 = b_j$, cross off the first row and go to the next step.

Step 3: Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied.

Example 1.3: Obtain the initial BFS to the following transportation problem using row minima and column minima method:

		Destination				
		D_1	D_2	D_3	D_4	Supply
Origin	O_1	6	3	5	4	22
	O_2	5	9	2	7	15
	O_3	5	7	8	6	8
	Demand	7	12	17	9	

Table 1.10: Example for row minimum method

Solution: In this methods, instead of finding the minimum cost cell as we did in the matrix minima method, we find the minimum cost cell in the first row or first column respectively. Then we allocate the maximum possible unit to that cell and proceed step by step deleting either a row or a column to get the shrunken matrix until all the rim conditions are satisfied.

Let us first apply the **Row Minima method** to the above described transportation problem. We first consider the first row O_1 in which minimum cost 3 is in the cell (1, 2). We allocate there the maximum possible units i.e., min(22, 12) = 12. As it satisfies all demands of D_2, so the column D_2 can be exhausted. Next, we allocate 9 units in the cell (1, 4) as it is the next minimum cost in this row. It satisfies all the demands of D_4. Next allocation is made in the cell (1, 3) with 1 unit. Then the availability of origin O_1 is exhausted.

Next we choose the second row and the allocate maximum units $15 = \min(15, 17)$ in the cell (2,3) as it has the minimum cost of transportation (i.e., 2) in this row. Next allocation is made in the third row's minimum cost cell i.e., in the (3, 1) cell with maximum units $7 = \min(8, 7)$. The last allocation is done in the next minimum cost of this row i.e., in the cell (3, 4) with the remaining 1 unit.

	D_1	D_2	D_3	D_4	a_i
O_1	6	[12] 3	[1] 5	[9] 4	22
O_2	5	9	[15] 2	7	15
O_3	[7] 5	7	[1] 8	6	8
b_j	7	12	17	9	

Table 1.11: Initial solution using row minima method

	D_1	D_2	D_3	D_4	a_i
O_1	6	[12] 3	[9] 5	[1] 4	22
O_2	[7] 5	9	[8] 2	7	15
O_3	5	7	8	[8] 6	8
b_j	7	12	17	9	

Table 1.12: Initial solution using column minima method

Thus the initial basic feasible solution of the problem is obtained as $x_{12} = 12$, $x_{13} = 1$, $x_{14} = 9$, $x_{23} = 15$, $x_{31} = 7$ and $x_{34} = 1$ which is shown in the Table 1.11. The number of allocation is 6, the solution is basic. The cost for this feasible solution is $12 \times 3 + 1 \times 5 + 9 \times 4 + 15 \times 2 + 7 \times 5 + 1 \times 6 = 143$.

In the similar way, we can apply the column minima method to this transportation problem and get the initial feasible solution which is shown in Table 1.12. From this table, we get the initial BFS as $x_{12} = 12$, $x_{13} = 9$, $x_{14} = 1$, $x_{21} = 7$, $x_{23} = 8$ and $x_{34} = 8$. The cost for this feasible solution is $12 \times 3 + 9 \times 5 + 1 \times 4 + 7 \times 5 + 8 \times 2 + 8 \times 6 = 184$.

3. Vogel's Approximation Method (VAM)

This method takes into account not only the least cost c_{ij} but also the costs that just exceed c_{ij}. The steps of the method are given below:

Step 1: For each row of the transportation table, select the smallest and next-to-smallest costs. Determine the difference between them for each row. These are called 'penalties'. Write down them alongside the transportation table enclosing by parentheses against the respective row. Similarly, compute these penalties for each column and write down against the respective column.

Step 2: Identify the row or column with the largest difference among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the largest difference be in the ith row and c_{ij} be the smallest cost in this row. Allocate the maximum possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross out the ith row or the jth column which is appropriate.

Step 3: Compute the row and column penalties for the reduced transportation table and go to step 2. Repeat the procedure until all the requirements are satisfied.

Example 1.4: Obtain the initial BFS of the following transportation problem using VAM:

		Destination				
		D_1	D_2	D_3	D_4	Supply
Origin	O_1	20	22	17	4	120
	O_2	24	37	9	7	70
	O_3	32	37	20	15	50
	Demand	60	40	30	110	

Table 1.13: Transportation table for Example 1.4

Solution: The penalties for rows and columns are shown in Table 1.14.

	D_1	D_2	D_3	D_4	a_i	Penalty
O_1	20	(40) 22	17	(80) 4	120	(13)(13)
O_2	(10) 24	37	(30) 9	(30) 7	70	(2)(2)(2)(17)(24)(24)
O_3	(50) 32	37	20	15	50	(5)(5)(5)(17)(32)
b_j	60	40	30	110		
Penalty	(4) (4) (8) (8) (8) (24)	(15)	(8) (8) (11)	(3) (3) (8) (8)		

Table 1.14: Initial BFS in VAM

The initial BFS by VAM is obtained as $x_{12} = 40$, $x_{14} = 80$, $x_{21} = 10$, $x_{23} = 30$, $x_{24} = 30$ and $x_{31} = 50$. The corresponding total transportation cost is $Rs.(40 \times 22 + 80 \times 4 + 10 \times 24 + 30 \times 9 + 30 \times 7 + 50 \times 32) = Rs.\ 3520$.